Topic 1

Properties of the straight line

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Learning Objectives

- Use the properties of the straight line

Minimum performance criteria:

- Determine the equation of a straight line given two points on the line or one point and the gradient
- Find the gradient of a straight line using \( m = \tan \theta \)
- Find the equation of a line parallel to and a line perpendicular to a given line
Prerequisites

A sound knowledge of the following subjects is required for this unit and this course in general:

- The symbols $\in$, $\notin$, $\emptyset$ in set theory
- The terms set, subset, empty set, member, element
- Number set convention:

  **STANDARD NUMBER SETS**

  where

  - $\mathbb{N}$ is the set of natural numbers, \{1, 2, 3, ...\}
  - $\mathbb{W}$ is the set of whole numbers, \{0, 1, 2, 3, ...\}
  - $\mathbb{Z}$ is the set of integers
  - $\mathbb{Q}$ is the set of rational numbers
  - $\mathbb{R}$ is the set of reals
1.1 Revision exercise

Learning Objective
Identify areas which need revision

Revision exercise

Q1: The following diagram represents a set relationship.

![Set Diagram]

a) Which is the universal set and in set notation write out its members?
b) Name the empty set.
c) How many elements are in set B?
d) Which set is a subset of another set?
e) Identify the elements of set C.
f) Is 4 a member of set B?

Q2: Identify the smallest number set \((\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{R})\) to which the following numbers belong:

a) \(\sqrt{2}\)
b) 0
c) 4
d) -3
e) \(\frac{2}{3}\)
f) -0.006

Q3: Enter the numbers and letters correctly in the following diagram using the information given.
1. The universal set = \{11, 13, 15, 17, 19, 21\}
2. A is a subset of B
3. C = \emptyset
4. 15 \in A
5. 13 \in B and D
6. 17, 19 and 21 are members of D

1.2 Gradients of straight lines

Learning Objective
Find the gradient of a straight line

The following straight lines drawn on the graph have one feature in common and yet they are all different.

They all cross the y-axis at the point (0, 2) but the slopes of the lines are different.

Gradient
The slope of a straight line from the point A \((x_1, y_1)\) to B \((x_2, y_2)\) is called the gradient.

It is denoted \(m_{AB}\) and defined as \(m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}\) where \(x_1 \neq x_2\)
1.2. GRADIENTS OF STRAIGHT LINES

A simple way of remembering the formula is

\[ m_{AB} = \frac{\text{change in } y}{\text{change in } x} \]

Example: Finding the gradient

Find the gradient of the following straight lines:

a) AB
b) PQ
c) RS
d) QP

Answer:

The gradients are:

a) \( m_{AB} = \frac{10 - 2}{-2 - (-10)} = \frac{8}{8} = 1 \)
b) \( m_{PQ} = \frac{-4 - 4}{-2 - 8} = \frac{-8}{-10} = \frac{4}{5} \)
c) \( m_{RS} = \frac{-8 - (-4)}{-2 - (-8)} = \frac{-4}{6} = -\frac{2}{3} \)
d) \( m_{QP} = \frac{4 - (-4)}{8 - (-2)} = \frac{8}{10} = \frac{4}{5} \)
Notice that $m_{PQ} = m_{QP}$

It does not matter which point is taken first.

It is however, essential that the x-coordinates are taken in the same order as the y-coordinates.

Gradients which have a plus value are called positive gradients and slope upwards from left to right. Negative gradients have a minus value and slope downwards from left to right.

There are two further sets of lines of particular interest:

- Those parallel to the x-axis and
- Those parallel to the y-axis

**Example : Gradient of a straight line parallel to the x-axis**

Find the gradient of the line AB shown:

\[
\text{Answer: } \quad m_{AB} = \frac{6 - 6}{6 - (-8)} = \frac{0}{14} = 0
\]

In fact, all lines parallel to the x-axis have a gradient of 0 since the top line of the equation will always be 0

**Example : Gradient of a straight line parallel to the y-axis**

Find the gradient of PQ in the following diagram:
Answer:

$$m_{PQ} = -\frac{2 - 6}{6 - 6} = -\frac{8}{0}$$

but this is undefined by the definition that $x_1 \neq x_2$
(dividing by zero is impossible).

This line has infinite gradient or has an undefined gradient.

All lines parallel to the y-axis have infinite gradient.

A further look at the gradient formula will give another useful way of calculating the gradient.
From the diagram \( \tan \theta = \frac{\text{change in } y}{\text{change in } x} \)

But \( m_{AB} = \frac{\text{change in } y}{\text{change in } x} \) and so \( m_{AB} = \tan \theta \)

**Gradient as a tangent**

The gradient of a straight line is the tangent of the angle made by the line and the positive direction of the x-axis. (assuming that the scales are equal).

If A is the point \((x_1, y_1)\) and B is the point \((x_2, y_2)\) then \( m_{AB} = \tan \theta \) where \( \theta \) is the angle made by the line and the positive direction of the x-axis.

**Example : Gradient of a straight line given the angle**

Find the gradient of AB:

Answer:

Angle BAC = angle BED and so \( \theta = 40^\circ \)

\( m_{AB} = \tan \theta = \tan 40^\circ = 0.8 \) to one decimal place.
1.2. GRADIENTS OF STRAIGHT LINES

Gradients of straight lines exercise

Q4: Find the gradients of the following straight lines:

a) AB
b) PQ
c) RS
d) CD

Q5: Find the gradient of the lines joining the points:

a) M (-4, 6) and N (3, 2)
b) L (-3, -3) and K (-3, 0)
c) P (2, -6) and Q (-4, -6)
d) T (4, -1) and the origin O.

Q6: Triangle ABC is an equilateral triangle with base AB on the x-axis and apex C above the x-axis. Find the gradient of each of the sides correct to 1 decimal place.

Q7: Draw a line through the point (2, 4) which has a gradient of -1 and state where it crosses the x and y axes.

Q8: Identify the gradients of the lines as: positive, negative, zero or infinite:
1.2.1 Distance between two points on a straight line

Learning Objective
Find the distance between two points on a straight line.

Given two points on a straight line, the gradient can be found but it can also be useful to know the distance between two points on the line.

By Pythagoras $AB^2 = AC^2 + BC^2$

Let A have coordinates $(x_1, y_1)$ and B have coordinates $(x_2, y_2)$

Then C has coordinates $(x_2, y_1)$

The Pythagoras equation becomes

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

and the length of AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example: Distance between two points

Find the distance between the points (3, 4) and (-2, 3)

Answer:

Using the formula,

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2 - 3)^2 + (3 - 4)^2} = \sqrt{26} = 5.1$ to 1 d.p.
Distance between two points exercise

Q9: Find the distance between the points correct to 1.d.p:

a) A (2, -1) and B (3, -5)
b) C (-2, -3) and D (1, 1)
c) E (-2, 0) and F (1, 1)

Calculator investigation and program

A program for a TI 83 graphics calculator follows which will provide questions on the gradients of straight lines. Try this if you have time and access to one.

To use the program calculate the gradient of the line shown, press enter, type in the gradient, press enter and the calculator will mark your answer.

Alternatively, investigate straight lines on a graphics calculator by plotting the equation in the form

\[ y = mx + c \]

for various values of \( m \) and \( c \)

```
:randInt(-5,5)⇒N
:Lbl 99
:"NX⇒Y1
:ZDecimal
:Pause
:Disp "GRADIENT="
:Input M
:If M=N
:Then
:Disp "WELL DONE"
:Else
:Disp "TRY AGAIN"
:Goto 99
```
1.3 Equation of a straight line

Learning Objective
Determine the equation of a straight line

A straight line can be identified uniquely from its gradient and the point at which it crosses the y-axis (called the y-intercept).

Look at the line in the diagram:

\[
\begin{align*}
\text{This line is the only line with gradient 2 which passes through the point (0, 1).}
\end{align*}
\]

Activity
Confirm that there are no other lines by trying to find one.

Equation of a straight line - standard form
The equation of a straight line takes the form \( y = mx + c \) where \( m \) is the gradient of the line and \( c \) is the y-intercept.

Examples

1. The equation of a straight line
Find the equation of the line with gradient -3 and which passes through the point (0, -4)
Answer:
The equation takes the form \( y = mx + c \)
\( m = -3 \) and \( c = -4 \) since the point (0, -4) is on the y-axis.
The equation is \( y = -3x - 4 \)

2. Find where the line \( 6x - 2y = 8 \) crosses the y-axis and state its gradient.
Answer:
\( 6x - 2y = 8 \iff 2y = 6x - 8 \iff y = 3x - 4 \)
Thus the gradient is 3 and it crosses the y-axis at -4, that is, at the point (0, -4)

3. The point P (a, 3) lies on the line y = 2x - 5, find the value of a

Answer:
Substitute y = 3 in the equation to give
3 = 2x - 5
2x = 8
x = 4
So when x = 4, y = 3 and the point P is (4, 3)
The value of a is 4

When a line is parallel to the x-axis the gradient is zero and the equation of the line becomes y = c where c is the y-intercept.

When a line is parallel to the y-axis the line does not cross y and the equation becomes x = k where k is value of x at the point where the line crosses the x-axis.

**Equation of a line as y = mx + c exercise**

**Q10:** Find the equation of the lines:

a) AB with gradient -4 and crossing the y-axis at y = -2
b) CD with gradient 0 and crossing the y-axis at 4
c) EF which is parallel to the y-axis and crosses the x-axis at x = 1
d) GH which has gradient -2 and passes through the point (0, -3)

**Q11:** State the gradients and y-intercepts of the lines:

a) y = -2x - 1
b) y = 3x
c) y = -5x - 5
d) x = 2
e) y = x - 1

The equation 6x - 2y = 8 was previously rearranged as y = 3x - 4 to represent the equation of a straight line.

This equation can also be rearranged as 6x - 2y - 8 = 0

These two equations are equal but give different forms of the equation of this straight line.

y = 3x - 4 is of the form y = mx + c

6x - 2y - 8 = 0 gives the general form of the equation of a straight line.

**Equation of a straight line - general form**

The general form of the equation of a straight line is

Ax + By + C = 0 (A, B not both zero).

Either form of the equation of a straight line can be used in calculations.
Conversion between equation forms exercise

Q12: Convert the following equations of a straight line into the general form and state the values for A, B and C:

a) \( y = -2x - 3 \)
b) \( y = 3x \)
c) \( y = 2x - 1 \)
d) \( y = 4 \)
e) \( x = -5 \)

Q13: Convert the following equations from the general form into the form \( y = mx + c \) and state the gradient and the y-intercept.

a) \( x - 4 = 0 \)
b) \( y + 3 = 0 \)
c) \( 2x + y - 3 = 0 \)
d) \( 6x - 3y + 9 = 0 \)

It is possible to find the equation of a straight line given either the gradient and one point on the line or alternatively given two points on the line.

The gradient of a line through two points \( A(x_1, y_1) \) and \( B(x, y) \) is given by

\[
m_{AB} = \frac{y - y_1}{x - x_1}
\]

Rearranging this gives \( y - y_1 = m(x - x_1) \)

If the gradient \( m \) and one of these points, say \( A \) is known then substitution of the values into the new equation will give the equation of the straight line. For example, if the gradient \( m = -2 \) and \( A \) is the point \((5, 3)\) then the equation becomes

\[
y - 3 = -2(x - 3) \Rightarrow y = -2x + 13
\]

This alternative formula is useful to remember.

**Equation of a straight line - alternative form**

Given one point and the gradient, the equation of the straight line can be found by using the formula \( y - y_1 = m(x - x_1) \) where \((x_1, y_1)\) is the known point and \( m \) is the gradient.

**Examples**

1. **The equation given one point and the gradient**

Find the equation of the line with gradient -2 and which passes through the point \((3, 4)\)

Answer:

\[
y - y_1 = m(x - x_1)
\]

Substituting the gradient of -2 and the point \((3, 4)\) will give \( y - 4 = -2(x - 3) \)

That is, \( y = -2x + 10 \)

The equation of the line is \( y = -2x + 10 \)
2. The equation given two points

Find the equation of a line which passes through the points (2, -1) and (4, 3)

Answer:

The equation takes the form $y = mx + c$

The gradient $m$ is

$$m = \frac{3 - (-1)}{4 - 2} = \frac{4}{2} = 2$$

Using the formula $y - y_1 = m(x - x_1)$ with the point (2, -1) gives $y + 1 = 2(x - 2) \Rightarrow y = 2x - 5$

Further equations exercise

**Q14:** Find the equations of the following lines:

a) AB which passes through the point (0, -2) and has gradient 4

b) GH which passes through the two points (3, 4) and (1, -2)

c) RS with gradient -4 and which passes through the point (2, -3)

d) CD which is parallel to the y-axis and passes through the point (-1, 0)

e) EF which passes through the points (0, 5) and (-3, 5)

In some cases, there is more than one correct answer to a question on straight lines. For example instead of finding one particular point which satisfies the information given, the answer may be the set of points on a particular straight line. In these cases it is the locus of the point which is required.

The term locus therefore is the collection of possible answers for a particular point or line.

**Example** Find the locus of the point A such that the length of AB = the length of AC where BC is a straight line.

Answer:

The locus of A is the straight line which is the perpendicular bisector of the line BC and not just a point on the line.
Any of the three forms (standard, general or alternative) of the equation of a straight line can be used if the correct information is given. The standard and alternative forms are used most. A quick reference on which formula to use dependant on the information known is given in the table:

<table>
<thead>
<tr>
<th>gradient</th>
<th>one point</th>
<th>y-intercept</th>
<th>EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>standard</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td></td>
<td>alternative</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>look again</td>
</tr>
</tbody>
</table>

In some cases, of course, preliminary work may be needed to find some of the information required such as the gradient.

1.4 Properties of the gradients of straight lines

**Learning Objective**

Use the properties of the gradients of parallel and perpendicular lines

**Q15:** Find the gradients of the two lines $y = 2x - 4$ and $y = 2x + 1$

Plot them on a graph. What relationships exist between the two lines?

The answer to the last question leads to an important property of the gradients of parallel lines.

Two distinct lines are parallel if and only if the lines have equal gradients.

**Example: Parallel lines**

Which of the sets of lines are parallel?

a) Line A where $y = -2x - 1$ and line B where $y = 2x + 1$

b) line AB where $A = (1, 2)$ and $B = (5, 6)$ and line PQ where $P = (-3, -1)$ and $Q = (-1, 1)$

Answer:

a) These are not parallel since line A has gradient -2 and line B has gradient 2

b) $m_{AB} = 1$, $m_{PQ} = 1$ The lines are parallel.

Note that if two lines have the same gradient and are not identified as distinct, inspection of the equation to find the y-intercept will reveal if they are in fact the same line.
1.4. PROPERTIES OF THE GRADIENTS OF STRAIGHT LINES

This relationship between parallel lines and equal gradients can be used to determine whether points are collinear, lie on parallel but distinct lines, or neither.

Example: Collinearity

Identify which sets of points are collinear.

a) A (2, 4), B (4, 8) and C (3, 6)

b) P (2, 3), Q (4, 5) and R (-1, -3)

c) K (-3, 2), L (-1, 1) and M (1, 1)

Answer:

a) \( m_{AB} = 2, m_{BC} = 2 \)

Since B is a common point on both lines and the gradients are the same the points are collinear. (Note: the gradients determine that the lines are parallel. The common point here in fact ensures that they are collinear.)

b) \( m_{PQ} = 1, m_{QR} = \frac{8}{5} \)

The gradients are not the same. The points are not collinear.
c) \( m_{KL} = -\frac{1}{2}, \ m_{LM} = 0 \)

The points are not collinear since the gradients of the two lines are not the same.

Now consider the lines L with equation \( y = \frac{1}{2}x - 3 \) and M with equation \( y = -2x + 4 \)

Q16: Plot the lines L with equation \( y = \frac{1}{2}x - 3 \) and M with equation \( y = -2x + 4 \) on a graph and determine the relationship between them.

The two lines in the last question are at right angles and there is a connection between this and their gradients giving another useful property of the gradients of straight lines.

Straight lines with gradients \( m_1 \) and \( m_2 \) are perpendicular if and only if \( m_1 m_2 = -1 \)
Example The gradients of a collection of lines are given. Which lines are perpendicular and which are parallel?

a) \( m_{AB} = -4 \)
b) \( m_{CD} = \frac{1}{4} \)
c) \( m_{EF} = 4 \)
d) \( m_{GH} = 2 \)
e) \( m_{IJ} = -2 \)
f) \( m_{KL} = \frac{1}{2} \)
g) \( m_{MN} = -\frac{1}{2} \)
h) \( m_{OP} = 4 \)

Answer:
The parallel lines are OP and EF
The perpendicular lines are AB and CD, GH and MN, IJ and KL

Perpendicular and parallel lines exercise

Q17: Find the gradient of the line perpendicular to the line through the points A (3, 5) and B (1, -3)

Q18: The gradient of PQ is 4.7
The line VW is perpendicular to PQ. Find the angle to the nearest degree that VW makes with the positive x-axis.

Q19: The vertices of a quadrilateral are A (2, 3), B (3, -1), C (8, 4) and D (6, 7) Are any of the lines parallel or at right angles and what shape is the quadrilateral?

Q20: A submarine is travelling from a point P with coordinates (-3, 7) to another point Q (3, 1). A frigate is on patrol on a bearing of 135° from the point R (-1, 1). Will the frigate pass over the path of the submarine? Explain.

1.5 Concurrency properties of straight lines in triangles

Learning Objective

Use the geometrical concurrency properties of straight lines in triangles

There are several interesting properties involving straight lines and triangles. These properties are concerned with the intersection of two or more lines at a point.

Concurrency

Lines which meet at a point are said to be concurrent.
1.5.1 Medians

Learning Objective
Know and use the properties of medians of a triangle.

A median of a triangle is a straight line from a vertex to the mid-point of the opposite side.

The medians of a triangle are concurrent and the point of intersection is called the centroid.

Construction activity
On graph paper construct a triangle with vertices A (2, 2), B (6, 6) and C (7, 2). Find and draw the medians. Confirm that the medians are concurrent.

The geometry is useful and clear to understand, but it is also important to understand the algebraic methods of finding the equation of a median and the point of intersection of the medians.

Strategy for finding the equation of a median

- Note the coordinates of the two end points of the line which the median meets.
- Find the midpoint coordinates of this line.
- Use the midpoint coordinates and the third vertex from where the median originates to determine the equation.

If the line AB has coordinates A \((a_1, a_2)\) and B \((b_1, b_2)\) the midpoint of this line is at the point with coordinates \(\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\right)\).

Example : The equation of a median
Find the equation of the median which touches the side AB in the triangle with vertices
at A (3, 0), B (-3, 2) and C (2, 6)

Let the point at which the median touches AB be D.

Then the median is the line CD.

The coordinates of D which is the midpoint of AB are

\[
\left( \frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2} \right) = \left( \frac{3 - 3}{2}, \frac{0 + 2}{2} \right) = (0, 1)
\]

The gradient of CD is

\[
\frac{y - 6}{x - 2} = \frac{5}{2}
\]

Using the equation of a straight line, \( y - y_1 = m(x - x_1) \) with the point C (2, 6) gives

\[
y - 6 = \frac{5}{2}(x - 2) \Rightarrow y = \frac{5}{2}x + 1
\]

(this can also be written in the general form of \( 5x - 2y + 1 = 0 \))

Remember: medians use midpoints. Midpoints are found from the two end points.

The point of intersection of the medians is very straightforward to find since it lies two thirds of the distance along each median measured from its vertex.

**Intersection point of the medians**

Let the points M (a, b), N (c, d) and P (e, f) be the vertices of a triangle. The point of intersection of the medians has coordinates

\[
\left( \frac{a + c + e}{3}, \frac{b + d + f}{3} \right)
\]

**Example : The point of intersection of the medians**

Find the point of intersection of the medians of a triangle which has vertices A (2, 2), B (6, 6) and C (7, 1)

Answer:

Using the equation for the point of intersection gives

\[
\left( \frac{2 + 6 + 7}{3}, \frac{2 + 6 + 1}{3} \right) = (5, 3)
\]

The point of intersection is (5, 3)

**Medians exercise**

**Q21**: Find the equations of the medians of a triangle which has vertices A (2, 2), B (6, 6) and C (7, 2)

**Q22**: Find the equations of the medians of the triangles with the following vertices:

a) A (3, 4), B (-3, 4) and C (1, 6)

b) P (-2, 0), Q (0, 4) and R (6, -2)
Q23: Find the equations of the medians and their point of intersection in the triangles with the following vertices:

a) C (1, 1), D(-3, -3) and E (5, -5)
b) K (-3, 1), L (1, -3) and M (1, 1)

Construction activity
The point of intersection of the medians can also be found by using the coordinates of the midpoints of the sides in the formula instead of the vertices of the triangle. Investigate this for a variety of triangles and confirm the result.

1.5.2 Altitudes
An altitude of a triangle is a straight line from a vertex perpendicular to the opposite side.

The altitudes of a triangle are concurrent and the point of intersection is called the orthocentre.

The equation of an altitude is straightforward to find using the property of perpendicular lines shown earlier.

Strategy for the equation of an altitude

- Find the gradient of the line that it meets.
- Use $m_1m_2 = -1$ to determine the gradient of the altitude.
- Use the coordinates of the third vertex from where the altitude drops and the gradient to determine the equation of the altitude.

Example: Equation of an altitude
Find the equation of the altitudes of the triangle with vertices A (2, -1), B (-3, -3) and C (-1, 2)

Answer:
Let the altitude from A to BC intersect BC at X
Let the altitude from B to AC intersect AC at Y
Let the altitude from C to AB intersect AB at Z

\[ m_{AB}, m_{CZ} = -1 \]
\[ m_{AC}, m_{BY} = -1 \]
\[ m_{BC}, m_{AX} = -1 \]

Since AB has gradient of \( \frac{2}{5} \) then the gradient of CZ is \( -\frac{5}{2} \)

By similar calculations, the gradient of BY is 1 and the gradient of AX is \( -\frac{2}{5} \)

The equation of CZ is \[ y - 2 = -\frac{5}{2}(x + 1) \]

That is, \[ y = -\frac{5}{2}x - \frac{1}{2} \]

The equation of BY is \[ y + 3 = x + 3 \]

That is, \[ y = x \]

The equation of AX is \[ y + 1 = -\frac{2}{5}(x - 2) \]

That is, \[ y = -\frac{2}{5}x - \frac{1}{5} \]

If the point of intersection of the altitudes is required this can be found by solving two equations for x or y and substituting this value in the third equation.

**Altitude exercise**

Q24: Find the equation of the altitudes in the triangles with vertices:

a) A (1, 4), B (-2, 5) and C (-1, 2)
b) K (2, 4), L (-1, -1) and M (3, 0)

**1.5.3 Bisectors of angles and sides**

There are two further concurrency properties.
The bisectors of the angles of a triangle are concurrent and the point of intersection is the centre of an inscribed circle.

The perpendicular bisectors of the sides of a triangle are concurrent and the point of intersection is the centre of a circumscribed circle.

Example: Equation of a perpendicular bisector

Find the equation of the perpendicular bisector of the line AB in the triangle ABC with vertices at A (-2, -2), B (4, -4) and C (0, 6)

Answer:
The gradient of AB is \(-\frac{1}{3}\)
The perpendicular bisector of AB has gradient 3 (using the property of perpendicular lines)
The mid point, say D, of AB is (1, -3)
The equation of the perpendicular bisector is therefore

\[ y + 3 = 3(x - 1) \]

That is, \( y = 3x - 6 \)
Bisector exercise

Q25: Find the equation of the perpendicular bisector of the line BC in the triangle with vertices A (-4, 3), B (0, 2), C (-5, -6).

State the equation in the general form $Ax + By + C = 0$

Q26: If the perpendicular bisector of the side AB of a triangle ABC has gradient of $-\frac{1}{2}$ and intersects AB at (3, -2) find the equation of the line AB

1.6 Summary

The following points and techniques should be familiar after studying this topic:

- Finding the gradient of a straight line.
- Finding the distance between two points.
- Using the different forms of the equation of a straight line dependent upon the information given.
- Using the properties of parallel and perpendicular lines.
- Using the properties of concurrency.

1.7 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest.

Archimedes

Archimedes was a Greek mathematician of the 3rd century whose ideas and discoveries led to many of the geometrical results known nowadays.

Euclid

He was another famous mathematician of the same era as Archimedes. He is best known for his work on 3 dimensional solids but, in The Elements he laid down some fundamental beliefs concerning points and straight lines.

d’Oresme

Oresme, though not as well known as some of the prominent mathematicians and scientists is considered to be the inventor of the coordinate system. It was his work which inspired Descartes.
Descartes
An 18th century mathematician who formalised the concepts of coordinate geometry and provided the link to the algebra behind it.

Newton
In the 17th century Newton developed a new coordinate system (polar coordinates). He is however well known in the field of calculus and had many arguments with another leading mathematician (Leibniz) over this.

1.8 Review exercise

Review exercise

This is an exercise which reflects the work covered at ‘C’ level in this topic.

Q27: The triangle PQR has vertices at P (-2, -4), Q (4, 3) and R (1, -3). Find the equation of the perpendicular bisector of PR

Q28: The line EF is at an angle of 35° with the positive x-axis and crosses it at the point (2, 0) Find the equation of EF

Q29: Find the equation of a line through the origin and parallel to the line through the points (2, 4) and (5, -1)

1.9 Advanced review exercise

Advanced review exercise

Q30: ABC is a right angled triangle with base AC parallel to the x-axis and vertices A (2, 3) and B (6, 8). Find the equation of the altitude CD from the vertex C to the side AB in the form Ax + By + C = 0

Q31: ABCD is a parallelogram.
The coordinates of A are (1, 2) and of B are (-1, -3) The gradient of AD is 5. Find

a) the equation of AD
b) the equation of AB
c) the equation of BC
d) If DC crosses y at y = -8 find the equation of DC
1.10 Set review exercise

Set review exercise

Work out your answers and then access the web to input your answers in the online test called set review exercise.

Q32: Find the equation of the line through the points (-3, 4) and (-5, 2) and state the equation of the line which intersects it at right angles at the y-intercept.

Q33: Find the intersection of the medians of the triangle with vertices A (-1, 0), B (0, 4) and C (2, 1)

Q34: The ship is at coordinates (-2, 7) and sees the lighthouse at coordinates (3, 4).
   a) If each unit is 1 nautical mile, how far away from the lighthouse is the ship?
   b) The ship is heading on a bearing of 125°. On what side of the ship will the lighthouse be when the ship passes it?
   c) Find the equation of the path of the ship.

Q35: A kite has a short diagonal with vertices at (-2, 3) and (5, 8)
   1. Find the equation of this diagonal.
   2. Hence find the equation of the leading diagonal.
Answers to questions and activities

1 Properties of the straight line

Revision exercise (page 3)

Q1:

a) \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \)

b) D

c) 4

d) A

e) \( C = \{1, 4, 5, 6, 7\} \)

f) yes

Q2: The smallest sets are as follows:

a) The set \( \mathbb{R} \)

b) The set \( \mathbb{W} \)

c) The set \( \mathbb{N} \)

d) The set \( \mathbb{Z} \)

e) The set \( \mathbb{Q} \)

f) The set \( \mathbb{I} \)

Remember that these answers identify the smallest set to which the particular number belongs.

Q3:

Gradients of straight lines exercise (page 9)

Q4:

a) \( m_{AB} = \frac{10 - 2}{2 + 10} = \frac{8}{12} = \frac{2}{3} \)
ANSWERS: TOPIC 1

b) \( m_{PQ} = \frac{-4 - 6}{-2 - 2} = \frac{-10}{-4} = \frac{5}{2} \)

c) \( m_{RS} = \frac{-8 - 2}{-2 + 8} = \frac{-10}{6} = -\frac{5}{3} \)

d) \( m_{CD} = 0 \) (parallel to x-axis)

Q5:

a) \( m_{MN} = \frac{2 - 6}{3 - (-4)} = -\frac{4}{7} \)

b) \( m_{LK} \) undefined or infinite (The line is parallel to the y-axis).

c) \( m_{PQ} = 0 \) (The line is parallel to the x-axis).

d) \( m_{TO} = \frac{-4}{-(-1)} = 4 \)

Q6: Draw a diagram:

```
\begin{array}{c}
\text{Y} \\
\text{A} \quad \text{B} \quad \text{C} \\
\text{X}
\end{array}
```

All angles = 60°

\( m_{AC} = \tan 60° = 1.7 \) to 1 decimal place.

\( m_{CB} = \tan (-60) = -1.7 \) to 1 decimal place.

\( m_{AB} = 0 \) (parallel to the x-axis).

Q7: If the gradient is -1 then \( \tan \theta = -1 \) where \( \theta \) is the angle between the line and the positive x-axis. So \( \theta = -45° \).
The line crosses the x-axis at (6, 0) and the y-axis at (0, 6)

**Q8:**
- AB has zero gradient
- CD has a negative gradient
- EF has an infinite gradient
- GH has a positive gradient

**Distance between two points exercise (page 11)**

**Q9:**

- **a)** \(\sqrt{(3 - 2)^2 + (- 5 + 1)^2} = \sqrt{17} = 4.1\) to 1 d.p.
- **b)** \(\sqrt{(1 + 2)^2 + (1 + 3)^2} = \sqrt{25} = 5.0\) to 1 d.p.
- **c)** \(\sqrt{(1 + 2)^2 + (1 - 0)^2} = \sqrt{10} = 3.2\) to 1 d.p.

**Equation of a line as y = mx + c exercise (page 13)**

**Q10:**

- **a)** AB has equation \(y = -4x - 2\)
- **b)** CD has equation \(y = 4\)
- **c)** EF has equation \(x = 1\)
- **d)** GH has equation \(y = -2x - 3\)

**Q11:**
a) The gradient is -2 and the y-intercept is -1
b) The gradient is 3 and the y-intercept is 0
c) The gradient is -5 and the y-intercept is -5
d) The gradient is undefined or infinite and there is no y-intercept
e) The gradient is 1 and the y-intercept is -1

Conversion between equation forms exercise (page 14)

Q12:

a) \(y + 2x + 3 = 0\) and \(A = 1, B = 2, C = 3\)
b) \(y - 3x = 0\) and \(A = 1, B = -3, C = 0\)
c) \(y - 2x + 1 = 0\) and \(A = 1, B = -2, C = 1\)
d) \(y - 4 = 0\) and \(A = 1, B = 0, C = -4\)
e) \(x + 5 = 0\) and \(A = 0, B = 1, C = 5\)

Q13:

a) This equation is \(x = 4\) and the line is parallel to the y-axis. It does not have a y-intercept.
b) \(y = -3\). This line has gradient zero and y-intercept of -3
c) \(y = -2x + 3\) with gradient = -2 and y-intercept of 3
d) \(y = 2x + 3\) with gradient of 2 and y-intercept of 3

Further equations exercise (page 15)

Q14:

a) The point (0, -2) is on the y-axis and so the y-intercept is -2
The gradient is 4 and so the equation of the line AB is \(y = 4x - 2\)
b) \(m_{GH} = \frac{-2 - 4}{1 - 3} = \frac{-6}{-2} = 3\)
The point (3, 4) is on the line and so the equation takes the form
\(y - 4 = 3(x - 3) \Rightarrow y = 3x - 5\)
The equation of the line GH is \(y = 3x - 5\)
c) RS has an equation of the form \(y - y_1 = -4(x - x_1)\)
The point (2, -3) lies on the line \(\Rightarrow y + 3 = -4(x - 2)\)
The equation of RS is \(y = -4x + 5\)
d) CD is parallel to the y-axis and so the equation takes the form \(x = k\)
Here \(k = -1\) since (-1, 0) is on the x-axis and the equation of CD is \(x = -1\)
e) EF has gradient of zero. Notice that the y co-ordinates are equal. This means that the line is parallel to the x-axis and the equation takes the form \(y = c\)
The point (0, 5) is on the y-axis and so the y-intercept is \(y = 5\)
The equation of EF is \(y = 5\)
Answers from page 16.

Q15: The gradient of \( y = 2x - 4 \) is 2
The gradient of \( y = 2x + 1 \) is 2
So the lines have the same gradient.
A graph shows that the lines are also parallel.

Answers from page 18.

Q16:

The two lines are perpendicular; that is they are at right angles.

Perpendicular and parallel lines exercise (page 19)

Q17: \( m_{AB} = 4 \)
If lines are perpendicular then the product of the gradients is -1
Let the gradient of the line perpendicular to AB be \( m_p \)
then \( m_p m_{AB} = -1 \Rightarrow m_p = -1/4 \)

**Q18:**

\[
\begin{align*}
\text{note that VW is any line} \\
\text{intersecting PQ at right angles} \\
\text{all such lines have the same} \\
\text{gradient \ldots they are parallel}
\end{align*}
\]

\[m_{PQ} = 4.7 \Rightarrow \text{PQ is at an angle of } \tan^{-1}(4.7)^\circ \text{ to the positive x-axis}\]

That is, at an angle of 78° to the nearest degree.

Thus if VW is perpendicular to PQ then VW is at an angle of 90 + 78 = 168° or -12° to the x-axis.

**Q19:**

\[m_{AB} = -4, \ m_{AD} = 1, \ m_{CD} = -3/2, \ m_{BC} = 1\]

The lines AD and BC are parallel. The shape is a trapezium.
Q20:

The gradient of the submarine path is -1
The submarine path makes an angle of -45° or 135° with the positive x-axis
\((\tan^{-1} (-1))\)
The frigate is on a bearing of 135°. This path makes an angle of 135° or -45° also with the x-axis.
The ship and the submarine are on parallel paths and the frigate will not cross the path of the submarine.

Medians exercise (page 21)

Q21: A median construction is from the vertex to the midpoint of the opposite side. The first step is therefore to find the midpoints of each side.

The mid point R of AB is \(\left(\frac{6 + 2}{2}, \frac{6 + 2}{2}\right) = (4, 4)\)
The mid point P of BC is \(\left(\frac{6 + 7}{2}, \frac{6 + 2}{2}\right) = \left(\frac{13}{2}, 4\right)\)
The mid point Q of AC is \( \left( \frac{2 + 7}{2}, \frac{2 + 2}{2} \right) = \left( \frac{9}{2}, 2 \right) \)

\[
m_{AP} = \frac{4 - 2}{13 - 2} = \frac{4}{9} \]

\[
m_{BQ} = \frac{2 - 6}{9 - 6} = \frac{8}{3} \]

\[
m_{CR} = \frac{4 - 2}{4 - 7} = -\frac{2}{3} \]

The equation of AP using the gradient just found and the point A is

\[
(y - 2) = \frac{4}{9}(x - 2) \]

\[
y = \frac{4}{9}x + \frac{10}{9} \]

The equation of BQ using the gradient just found and the point B is

\[
(y - 6) = \frac{8}{3}(x - 6) \]

\[
y = \frac{8}{3}x - 10 \]

The equation of CR using the gradient just found and the point C is

\[
(y - 2) = -\frac{2}{3}(x - 7) \]

\[
y = -\frac{2}{3}x + \frac{20}{3} \]

Q22:

a) Let V be the midpoint of AB, W the midpoint of BC and X the midpoint of AC
then the coordinates of these points are
V (0, 4), W (-1, 5) and X (2, 5)

Using the gradient formula gives

\[
m_{AW} = -\frac{1}{4} \]

\[
m_{BX} = \frac{1}{5} \]

\[
m_{CV} = 2 \]

Using the alternative equation of a straight line \( y - y_1 = m(x - x_1) \) for each median
in turn gives the equations:

\[
AW : y = -\frac{1}{4}x + \frac{19}{4} \]

\[
BX : y = \frac{1}{5}x + \frac{23}{5} \]

\[
CV : y = 2x + 4 \]

b) Let the midpoints be I for PQ, J for line QR and K for line PR
line IR has equation \( y = -\frac{4}{7}x + \frac{10}{7} \)
line JP has equation \( y = \frac{1}{5}x + \frac{2}{5} \)
line KQ has equation \( y = -\frac{5}{2}x + 4 \)
Q23:

a) Let V be the midpoint of CD, W the midpoint of DE and X the midpoint of CE then the coordinates of these points are:
V = (-1, -1), W = (1, -4) and X = (3, -2)
The equations of the medians are:
CW: \( x = 1 \) (note that the gradient is undefined and so the median is parallel to the y-axis. The two points indicate that \( x = 1 \))
DX: \( y = \frac{1}{6}x - \frac{5}{2} \)
CW: \( y = \frac{-2}{3}x - \frac{5}{3} \)
The point of intersection is
\[
\left( \frac{1 \cdot 3 + 5}{3} , \frac{-3 \cdot 5}{3} \right) = \left( 1, \ -\frac{7}{3} \right)
\]

b) Let X be the midpoint of KM, Y be the midpoint of KL and Z be the midpoint of LM These points have coordinates: X = (-1, 1), Y = (-1, -1) and Z = (1, -1)
The equations of the medians are:
XL: \( y = -2x - 1 \)
YM: \( y = x \)
KZ: \( y = -\frac{1}{2}x - \frac{1}{2} \)
The point of intersection of the medians is \( \left( \frac{1}{3} , \ -\frac{1}{3} \right) \)

Altitude exercise (page 23)

Q24:

a) Let the points of intersection of the altitudes with AB, AC and BC be X, Y and Z respectively then
AZ has gradient \( \frac{1}{3} \), BY has gradient -1, CX has gradient 3
The equations of the altitudes are:
AZ: \( y = \frac{1}{3}x + \frac{11}{3} \)
BY: \( y = -x + 3 \)
CX: \( y = 3x + 5 \)

b) Let the points of intersection of the altitudes with LM, KM and LK be X, Y and Z respectively then
KX has gradient -4, LY has gradient \( \frac{1}{4} \), MZ has gradient \( \frac{-3}{5} \)
Note that this is a right angled triangle and some work can be saved by noticing that the altitude LY is in fact the side LM and similarly the altitude KY is the side KM
The equations of the altitudes are:
KX: \( y = -4x + 12 \)
LY: \( y = \frac{1}{4}x - \frac{3}{4} \)
MZ: \( y = \frac{-3}{5}x + \frac{9}{5} \)
**Bisector exercise (page 25)**

**Q25:** The mid point of BC is \((-\frac{5}{2}, -2)\)

- The gradient of BC is \(\frac{8}{5}\) and so the perpendicular bisector has gradient \(-\frac{5}{8}\)
- The equation of the perpendicular bisector is \(y + 2 = -\frac{5}{8}(x + \frac{5}{2})\)
- That is, \(16y + 10x + 57 = 0\)

**Q26:** The side AB has gradient 2 since the product of the gradients of perpendicular lines is -1.

- The equation of the side AB is \(y + 2 = 2(x - 3)\) since (3, -2) is on the line.
- That is the equation of AB is \(y = 2x - 8\)

**Review exercise (page 26)**

**Q27:** The mid point of PR is \((-\frac{1}{2}, -\frac{7}{2})\)

- The gradient of PR is \(\frac{1}{3}\) and so the gradient of the perpendicular bisector is -3
- Thus the equation is \(y + \frac{7}{2} = -3(x + \frac{1}{2})\)
- That is, \(y = -3x - 5\)

**Q28:** \(m_{EF} = \tan 35^\circ = 0.7\)

- The equation is \(y = 0.7(x - 2)\)
- That is, \(y = 0.7x - 1.4\) or \(10y - 7x + 14 = 0\)

**Q29:** The gradient of the line through the two points is \(-\frac{5}{3}\)

- The line parallel to this line through the origin has a y-intercept of 0 and so has the equation \(y = -\frac{5}{3}x\)

**Advanced review exercise (page 26)**

**Q30:** C is the point (6, 3)

- The gradient of AB is \(\frac{5}{4}\) and so the gradient of the altitude to AB is \(-\frac{4}{5}\)
- The equation of the altitude is \(y - 3 = -\frac{4}{5}(x - 6)\)
- That is, \(5y - 15 = -4(x - 6)\)
- Thus \(5y + 4x - 39 = 0\)

**Q31:**

a) The equation of AD is \(y - 2 = 5(x - 1)\) that is \(y = 5x - 3\)

b) The gradient of AB is \(\frac{5}{2}\)
- Thus the equation of AB is \(y - 2 = \frac{5}{2}(x - 1)\)
- That is \(2y = 5x - 1\)

c) BC is parallel to AD and so has gradient 5
- The equation of BC is \(y + 3 = 5(x + 1)\)
- That is \(y = 5x + 2\)
Since AB is parallel to CD then CD has gradient $\frac{5}{2}$ and has an equation of the form $2y = 5x + c$
but $y = -8$ when $x = 0$ and so $c = -16$
Thus the equation is $2y = 5x - 16$

**Set review exercise (page 27)**

**Q32:** The answers are only available on the web.

**Q33:** The answer is only available on the web.

**Q34:** The answers are only available on the web.

**Q35:** The answers are only available on the web.